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CONTRACTION AND FIXED POINT BEHAVIOR OF CERTAIN LINEAR
FRACTIONAL TRANSFORMATIONS

A Project
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Mathematics

by
Haragewen Abraham Kinde

December 1992


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Abstract

Problem: What happens when we apply the theory of contraction maps on the unit circle, in the complex plane, to linear fractional transformations of monomial and triangular types.

Method and Design: See Page 3.

Conclusions: We examine the relation between;

- 1) analyticity on the unit disk,
- 2) contraction behavior on the unit circle,
- 3) fixed point behavior on the unit disk. To draw conclusions about the action of some important subgroups of linear fractional transformations.

Table Of Contents

1. Introduction	1
2. General Aims and Strategies	3
3. Basic Results From Complex Analysis	5
4. Contraction And Fixed Point Behavior Of Certain Linear Fractional Transformations	8
5. Conclusion	19

APPENDICES

A. The Invariant σ	22
B. Computer Images	30
C. Supplemental Information	36
BIBLIOGRAPHY	38

1. Introduction

The purpose of this project is to apply the theory of contraction maps on the unit circle, in the complex plane, to linear fractional transformations of monomial and triangular type.

A linear fractional transformation T is a rational function of the form $T(z) = \frac{az + b}{cz + d}$, where a, b, c , and d are complex numbers and $ad - bc \neq 0$. This restriction is very essential, for otherwise $T'(z) = \frac{ad - bc}{(cz + d)^2} = 0$ for all z , so T is identically constant. The function T maps distinct points onto distinct images. Note also that T has a pole of order one at $-d/c$ and $\lim_{|z| \rightarrow \infty} T(z) = a/c$.

Hence, a linear fractional transformation is a one - to - one mapping of the complex plane plus the point at ∞ onto itself. Conversely, a one - to - one function (analytic) mapping of the complex plane plus " ∞ " onto itself is a linear fractional transformation. In addition, a linear fractional transformation that is not identically equal to z has, at most, two distinct fixed points z for which $T(z) = z$.

Linear fractional transformations are very important in the study of mathematics. Some of the properties of these

transformations are : a) They are functions that preserve angles between curves. Hence, they are important tools in studying flows and fields, and in solving boundary - value problems. b) Since linear fractional transformations can be factored into translations, rotations, dilatations, and reciprocation, (which are called "simple types") these transformations are extremely important in the study of geometry. c) Linear fractional transformations have a major role in the study of real one - dimensional projectivities, since every projectivity can be represented by a linear fractional transformation. Finally, a geometrical characterization of linear fractional transformations is that they are the only circle preserving transformations in the completed plane which also preserve orientation.

2. General Aims And Strategies

We will prove and make use of two basic lemmas. The first one is from complex analysis and is a consequence of Rouché's Theorem. The second one is a computational lemma from the theory of linear fractional transformations.

We will use T to denote an arbitrary linear fractional transformation. The set of all transformation T forms a group under composition which we will denote by G . It is well known that G is isomorphic to the projective linear group over complex numbers.

We will analyze the action of transformations in the monomial subgroup M of G consisting of those T with either $a = d = 0$ or $b = c = 0$, and of transformations in the upper and lower triangular subgroups of G . We denote the upper and lower triangular subgroups by U and L respectively.

In addition, throughout this text we will let $C = \{ z : |z| = 1 \}$ the unit circle, $D = \{ z : |z| < 1 \}$ the open unit disk. We will be concerned with images of C and D under a transformation T . By way of definition, we say that T contracts, or shrinks C provided $|T(\exp(i\theta))| < 1$ for all θ . In particular, we will study relations among three

important properties of these actions:

- 1) Analyticity on the unit disk,
- 2) Contraction behavior on the unit circle,
- 3) Fixed Point behavior on the unit disk.

The reason for focusing on these particular subgroups is that certain surprising conclusions will be reached regarding the interplay of these three properties. These conclusions will be stronger than the lemmas we use for tools, and will be more specific.

Finally, we note that elements of the group G are classified up to similarity by an invariant known as the magnitude of T ; see[3]. This invariant determines whether the transformation T is of Elliptic, Parabolic, Proper Hyperbolic, Improper Hyperbolic, and Loxodromic type. In appendix A , we will provide examples of transformations exhibiting the behaviors we have analyzed in this project according to the various similarity type. In addition, we will illustrate some of these conclusions using computer pictures of images of the disk for some examples in Appendix B.

3. Basic Results From Complex Analysis

Throughout this exposition we will need certain result from complex analysis. One of the important results is Rouche's Theorem, which is as follows: Suppose f and g are analytic on an open set containing a piecewise smooth simple closed curve Γ and its inside. If $|f(z) + g(z)| < |f(z)|$ for all $z \in \Gamma$, then f and g have an equal number of zeros inside Γ , counting multiplicities. (The proof of this theorem can be found in any complex variable text, such as [2].)

Lemma(1) : Suppose that f is analytic on a domain containing $\{ z : |z| \leq 1 \}$ and that $|f(\exp(i\theta))| < 1$ for $0 \leq \theta \leq 2\pi$. Then f has exactly one fixed point in the disk $|z| < 1$; that is, if f shrinks C then the equation $f(z) = z$ has precisely one solution in D .

Proof:

Let $g(z) = f(z) - z$ and $h(z) = z$. Clearly both g and h are analytic on a domain containing the closed unit disk. Further, $|g(\exp(i\theta)) + h(\exp(i\theta))| = |f(\exp(i\theta))| < 1 = |h(\exp(i\theta))|$ for all θ , $0 \leq \theta \leq 2\pi$. Hence, by Rouche's Theorem g and h have the same number of zeros inside C . In other words, g has exactly one zero inside C . And so f has a unique fixed

point in D counting multiplicities. //

(We will refer to Lemma(1) as the fixed point lemma.)

We will also make extensive use of the following technical lemma which is purely a consequence of the complex arithmetic of Linear Fractional Transformations.

Lemma(2): A Linear Fractional Transformation $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ contracts C iff $|a|^2 + |b|^2 < |c|^2 + |d|^2 - 2|c\bar{d} - a\bar{b}|$.

Proof:

$|T(\exp(i\theta))| = \frac{|a(\exp(i\theta)) + b|}{|c(\exp(i\theta)) + d|}$ and this fraction is less than one iff $|a(\exp(i\theta)) + b| < |c(\exp(i\theta)) + d|$. Since both sides are non-negative, this inequality is equivalent to $|a(\exp(i\theta)) + b|^2 < |c(\exp(i\theta)) + d|^2$. Since $|z|^2 = z\bar{z}$, we have

$$(\bar{a}(\exp(-i\theta)) + \bar{b})(a(\exp(i\theta)) + b) < (\bar{c}(\exp(-i\theta)) + \bar{d})(c(\exp(i\theta)) + d)$$

which yields

$$a\bar{a} + b\bar{b} + a\bar{b}(\exp(i\theta)) + b\bar{a}(\exp(-i\theta)) < c\bar{c} + d\bar{d} + c\bar{d}(\exp(i\theta)) + d\bar{c}(\exp(-i\theta))$$

and then

$$|a|^2 + |b|^2 + 2\operatorname{Re}(a\bar{b}(\exp(i\theta))) < |c|^2 + |d|^2 + 2\operatorname{Re}(c\bar{d}(\exp(i\theta)))$$

which we can write as

$$|a|^2 + |b|^2 < |c|^2 + |d|^2 + 2\operatorname{Re}(\exp(i\theta))(c\bar{d} - a\bar{b})$$

Since only the far right hand term depends on θ and the inequality must hold for all θ , the contraction property will hold iff the inequality is satisfied when the far right hand term achieves its minimum . However, the minimum value of this term is $-2|c\bar{d} - a\bar{b}|$. Therefore, $|T(\exp(i\theta))| < 1$ for all θ iff $|a|^2 + |b|^2 < |c|^2 + |d|^2 - 2|c\bar{d} - a\bar{b}|$. //

(We will refer to Lemma(2) as the contraction lemma.)

In order to demonstrate the usefulness of the contraction lemma, note that a transformation of the form $T = z / (z+d)$ shrinks C iff $|d| > 2$.

4. Contraction And Fixed Point Behavior Of Certain Linear Fractional Transformations

A. Monomial Transformations : These are transformations in the subgroup $M = \{ T \in M : b = c = 0 \text{ or } a = d = 0 \}$.

Note that the set of transformations $T = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ form a

a subgroup of index 2 in M . The other coset consists of those transformations $T = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$. Thus, it is clear

why they are called monomial transformations. They have either the form $T(z) = \frac{b}{c(z)}$ or $T(z) = \frac{a(z)}{d}$.

Case I. $a = d = 0$, $T(z) = b / c(z)$.

If $|b| < |c|$ then

1. T has singularity at the origin.
2. T shrinks C .
3. T has two fixed points inside D .

Example: $T(z) = \frac{i}{(4+i)z}$; fixed points are $\pm(1+4i)^{\frac{1}{2}} / (17)^{\frac{1}{2}}$ and property (2) is satisfied by the contraction lemma.

If $|b| = |c|$ then

1. T has singularity at the origin.
2. T does not shrink C .
3. T has two fixed points on C .

Example: $T(z) = 1/z$; fixed points are ± 1 which is on the

unit circle and property (2) is satisfied by the contraction lemma.

If $|b| > |c|$ then

1. T has singularity at the origin.
2. T does not shrink C .
3. T has two fixed points outside D .

Example: $T(z) = 7 / 2z$; fixed points are $\pm (7/2)^{\frac{1}{2}}$ and property (2) is satisfied by the contraction lemma.

Case II. $b = c = 0$, $T(z) = a(z) / d$.

1. T is analytic in D .
2. T shrinks C iff $|a| < |d|$.
3. T has zero as its only fixed point in D .

Example: $T(z) = z / (4+i)$; the unique fixed point is zero and by contraction lemma T shrinks C .

Note: The next two follow easily by the contraction lemma.

i) If $|a| = |d|$ we have a pure rotation on $T(z) = \exp(i\theta)z$. Thus, T does not shrink C . ii) If $|a| > |d|$ T does not shrink C .

B. Upper Triangular Transformations : These are transformations in the subgroup $U = \{ T \in G : c = 0 \}$.

These transformations $T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ have the form $T(z) = \frac{az+b}{d}$.

Sometimes they are referred to as integral transformations.

I. Suppose $a = d$, then $T(z) = z + b/d$. Here we have transformations of the form $T(z) = z + \Gamma$ where,

1. T is analytic in D .
2. T does not shrink C for any Γ
3. T has no fixed points unless $\Gamma = 0$, which implies T equals the identity function which fixes everything.

Geometrically T is a pure translation which moves the unit circle to a circle of radius 1 centered at $\Gamma = b/d$. Let us use the contraction lemma to give an analytic proof of property (2).

Proof of property(2):

$T(z) = z + \Gamma$. Assuming Γ not equal to zero and applying the contraction lemma to T , if T shrinks C , we would have

$$1 + |\Gamma|^2 < 1 - 2|-\bar{\Gamma}|$$

$$|\Gamma|^2 < -2|\Gamma|$$

$$|\Gamma| < -2, \text{ a contradiction.}$$

Since $\Gamma = b/d$, Γ could be any non-zero complex number.

Thus, T does not shrink C . //

II. Suppose $b = d$, then $T(z) = \Gamma z + 1$. Here, as $\Gamma = a/d$ varies, the images of C under T form a family of concentric circles centered at 1. In addition,

1. T is analytic in D .
2. T does not shrink C for any Γ .
3. If $\Gamma \neq 1$, T has unique fixed point which may or may not be in the unit disk. If $\Gamma = 1$, this implies, T has no fixed point.

Proof of property (2):

$T(z) = \Gamma z + 1$ once again, assuming Γ not equal to zero and applying the contraction lemma to T , if T shrinks C , we would have

$$|\Gamma|^2 + 1 < 1 - 2|\Gamma|$$

$$|\Gamma|^2 < -2|\Gamma|$$

$$|\Gamma| < -2, \text{ a contradiction.}$$

Since $\Gamma = a/d$, Γ could be any non-zero complex number. Thus, T does not shrink C . //

Note: This case, $b = d$ and $c = 0$, is a counter example to the converse of fixed point lemma. Which is: If T has a unique fixed point in C and is analytic in D , then T shrinks C .

Example: Let $\Gamma = 3$, $T(z) = 3z + 1$,

1. T is analytic in D .

2. T does not shrink C by contraction lemma.

3. T has unique fixed point in D . Which is $-\frac{1}{2}$.

So far we have looked at specific cases. Now we will conclude with the general case. That is the case where a , b and d are possibly distinct.

III. Suppose a , b and d are arbitrary. Then $T(z) = \frac{az + b}{d}$ where

1. T is analytic in D .

2. T shrinks C iff $|b| < |d| - |a|$.

3. T has unique fixed point in D iff $|b| < |d - a|$.

Proof of property (2) :

Once again by the contraction lemma

$|a|^2 + |b|^2 < |c|^2 + |d|^2 - 2|c\bar{d} - a\bar{b}|$. But since $c = 0$ we have

$$|a|^2 + |b|^2 < |d|^2 - 2|a\bar{b}|$$

$$|a|^2 + 2|a||b| + |b|^2 < |d|^2$$

$(|a| + |b|)^2 < |d|^2$ since both expressions are positive we can take their square root to obtain,

$$|a| + |b| < |d|$$

$$|b| < |d| - |a|. \quad //$$

Note: We have actually verified the fixed point lemma for the case a , b , and d arbitrary and $c = 0$, without the use of

Rouche's Theorem. We contrast the following two examples:

Example(1):

$$T(z) = ((1+i)z + i) / (5 - 2i) ,$$

Example(2):

$$T(z) = \frac{(5 - 2i)z - i}{1+i} \quad \text{in both cases, the fixed point}$$

$(-3 + 4i) / 25$ is in D . In the first case T shrank C , where as in the second case T does not shrink C .

C. Lower Triangular Transformations: These are transformations in the subgroup $L = \{ T \in G : b = 0 \}$. They are of the form $T(z) = az / (cz + d)$ we have

1. T has singularity at $-d/c$.
2. T shrinks C iff $|a|^2 < (|d| - |c|)^2$.
3. T has unique fixed point in D iff $|c| < |a-d|$.

We will need to prove properties (2) and (3). But before doing so we will show directly, without the use of Rouche's Theorem, that property (2) implies property (3) provided T is analytic in D .

Proof:

Assume $|c| < |d|$. From $|a|^2 < (|d| - |c|)^2$ we can conclude

$$|a| < |d| - |c|$$

which implies $|c| < |d| - |a|$

using the Triangle Inequality we get

$$|c| < |d - a|$$

Thus, T has unique fixed point in D . //

Now we will prove properties (2) and (3).

Proof of property (2) :

$T(z) = (az) / (cz + d)$. Using contraction lemma directly we get

$$|a|^2 < |c|^2 + |d|^2 - 2|cd|$$

$$|a|^2 < (|d| - |c|)^2$$

Therefore, T shrinks C iff $|a|^2 < (|d| - |c|)^2$. //

Proof of property (3) :

$T(z) = (az) / (cz+d)$. Note, for a transformation in this group, zero will always be a fixed point. So, in order to find the other fixed point, set $T(z) = z$. Now we have

$$(az) / (cz + d) = z$$

$$az = cz^2 + dz$$

$$cz^2 + (d-a)z = 0$$

$$z(cz + (d-a)) = 0$$

either $z = 0$ or $cz + (d-a) = 0$. Thus, the other fixed point is $z = (a-d) / c$. Therefore, T will have a unique fixed point in D iff $|a-d| / |c|$ is greater than one, which implies $|c| < |a-d|$. //

Since the transformations in L have singularity at $-d/c$, we investigate further to determine the behavior of T when the pole is inside or outside D . Thus, we observe the following:

Case I. $|c| < |d|$ analytic in D .

By the contraction lemma if T shrinks C then T has unique fixed point in D .

Example: $T(z) = \frac{iz}{(2+i)z + 7}$.

By the contraction lemma T shrinks C and T has two fixed points, 0 and -3, where 0 is the unique fixed point in D . (see Appendix B, Fig. 1, for image)

On the other hand, if T does not shrink C , T may or may not have unique fixed point in D .

Example: $T(z) = \frac{10z}{z + 6}$.

By the contraction lemma T does not shrink C , and T

has two fixed points, 0 and 4, where 0 is the unique fixed point in D.

Example: $T(z) = \frac{3z}{z + 3}$.

Once again by the contraction lemma T does not shrink C, and T has a fixed point of multiplicity two at the origin.

Note: The case where T shrinks C and T has no unique fixed point cannot occur since property (2) implies property (3).

Case II. $|c| > |d|$ singularity inside D.

By contrast with case I, if T shrinks C it will necessarily have two fixed points in D. We see this as follows:

Proof:

If zero was the only fixed point in D we would have

$$|c| < |a-d|$$

which is

$$< |a| + |d|.$$

But then

$$|c| - |d| < |a|$$

which is equivalent to

$$|a|^2 > (|c| - |d|)^2.$$

However, this final inequality implies that T cannot shrink C . //

Since this result is analogous to our fixed point lemma for analytic maps, we summarized it in the form of a theorem.

Theorem : Let $T \in L$, with its pole inside D . If T has a fixed point outside of D then T cannot shrink C . (See Appendix C for supplemental information.)

The other possibilities for property (2) and (3) can occur, as the following examples show.

EITHER:

By the contraction lemma T does not shrink C , and T has unique fixed point in D .

Example: $T(z) = \frac{-7z}{(2+i)z - i}.$

T has two fixed points 0 and $3/2$, one inside one outside of D . Hence, 0 is the unique fixed point. (see Appendix B, Fig. 2, for image)

OR:

By the contraction lemma T shrinks C , and T does not have unique fixed point in D .

Example: $T(z) = \frac{iz}{7z + (2+i)}.$

T has two fixed points 0 and $-2/7$. Hence, both fixed points are inside D. (see Appendix B, fig. 3, for image)

OR:

The following examples show the question of multiplicity is crucial for property (3).

By the contraction lemma T shrinks C, and T does not have unique fixed point in D.

Example: $T(z) = \frac{z}{4z + 1}.$

T has a fixed point of multiplicity two at the origin.

OR:

By the contraction lemma T does not shrink C, and T does not have unique fixed point in D.

Example: $T(z) = \frac{2z}{3z+2}.$

T has a fixed point of multiplicity two at the origin.

5. Conclusion

In conclusion , we have used linear fractional transformations monomial and triangular types as explication of the fixed point and contraction lemmas from complex analysis. However, we have seen some surprising situations and have cited examples for these situations. In short summarizing the different classes we have the following:

I. In the class of the monomial transformations we observed that EITHER T had singularity at the origin where T a) shrank C and had two fixed points in D , b) did not shrink C and had two fixed points in D , c) did not shrink C and the two fixed points were outside D ; OR T was analytic in C where T a) shrank C and had unique fixed point in D , b) did not shrink C and had no unique fixed point in D . Hence, a pure rotation. Thus, part a) of the latter conclusion showed that this was the only situation where the fixed point lemma could be satisfied under this class.

II. In the class of the upper triangular transformations we observed that : T was analytic in D where T a) did not shrink C for any Γ and had no fixed point in D , b) did not shrink C for any Γ and had unique fixed point which may or may not be in D . Notice, this situation implies

that the converse of the fixed point lemma will not hold. (Surprising!!) c) did shrink C iff $|b| < |d| - |a|$ and has unique fixed point iff $|b| < |d - a|$. Notice this conclusion proves the fixed point lemma without Rouché's Theorem. This is another surprising outcome!!

III. In the class of the lower triangular transformations we observed that: a) T had singularity at $-d/c$, b) T shrank C iff $|a|^2 < (|d| - |c|)^2$, c) T had unique fixed point in D iff $|c| < |a - d|$. In addition b) and c) were disjoint from one another. (This was an interesting conclusion !)

Due to the singularity at $-d/c$ we were curious to find out about the behavior of T when the pole was inside or outside of D . Thus, after several investigations we concluded that EITHER : T was analytic in D where T a) shrank C and had unique fixed point in D , b) did not shrink C and had unique fixed point in D . OR: T had singularity inside D where T a) did not shrink C and had the origin as its unique fixed point in D , b) shrank C and had two fixed points inside D , c) did shrink C and had a fixed point of multiplicity two at the origin d) did not shrink C and had a fixed point of multiplicity two at the origin.

Thus, as a result of these surprising outcomes for this class, we were able to summarize our observations in a theorem.

Theorem : Let $T \in L$ with its pole inside of D . If T has a fixed point outside of D then T can not shrink C .

Appendix A

The Invariant σ

We have investigated the manner in which the monomial and triangular type linear fractional transformations behave with respect to the three properties. That is : 1) analyticity on the unit disk, 2) contraction behavior on the unit circle, 3) fixed point behavior on the unit disk. We shall now study, using tables, their similarity invariant defined by ; $\sigma = \frac{(\text{tr } T)^2}{|T|} - 4$ where $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

σ is sometimes called the magnitude of the linear fractional transformation. Note that it can be shown that two linear fractional transformations are similar in the group G iff their magnitudes are equal. See [3]

The transformations T are classified as:

Parabolic { which is similar to a translation } if $\sigma = 0$,

Elliptic { similar to a rotation } if $-4 \leq \sigma < 0$,

Proper Hyperbolic {similar to a dilatation} if $\sigma > 0$,

Improper Hyperbolic {similar to a dilatation} if $\sigma \leq -4$,

Loxodromic {leaves no circle invariant} if σ is not real.

Once again, [3] describes this choice of terminology in terms of invariant pencils of circles in the Moebius plane.

Note: for the triangular type we have either :

$$T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \quad \text{or} \quad T = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} .$$

Note that, $\sigma = \frac{(a + d)^2}{ad} - 4$ in both cases.

For the monomial type there is the additional possibility

$$T = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \quad \text{in which case } \sigma = -4.$$

In this appendix we will indicate whether or not a transformation of a particular type can exhibit the properties (1), (2), and (3), which we have analyzed in this paper. For example: if we were considering $T \in M$ then we will find neither Parabolic nor Proper Hyperbolic nor Loxodromic transformations which are analytic in D . In this case there are no transformations of these type in M . We will indicate this in the table by the word NP to imply not possible. On the other hand M contains Improper Hyperbolic transformations . However, they are singular at the origin. We will indicate this in the table by the word No.

Similarity Invariant Tables

Monomial Type :

I. $T(z) = b / cz$, $a = d = 0$. In this case σ is always equal to -4 , whether $|b|=|c|$ or $|b|<|c|$ or $|b|>|c|$. Thus, T is similar to an Elliptic or Improper Hyperbolic Transformation.

Example:

$$T(z) = \frac{i}{(4+i)z}, \quad \sigma = -4$$

Table 1 : Similarity Invariant Behavior of M Case I.

Transformations	Analytic in D	Shrinks C	! Fixed Point
Elliptic	No	No	Yes
Parabolic	NP	NP	NP
Prop. Hyperbolic	NP	NP	NP
Impr. Hyperbolic	No	No	Yes
Loxodromic	NP	NP	NP

II. $T(z) = az / d$, $b = c = 0$,

a) case $|a| = |d|$;

Example : $T(z) = z$, $\sigma = 0$, in this case T is similar to a Parabolic Transformation , has no unique fixed point , is analytic in D and T does not shrink C.

b) case $|a| < |b|$;

In this case T could either be similar to a Proper Hyperbolic Transformation, which implies that $\sigma > 0$,

Example : $T(z) = (2i)z / 5i$, $\sigma = 0.9$.

or T could be similar to an Improper Hyperbolic Transformation . This implies that $\sigma \leq -4$,

Example : $T(z) = (3i)z / -4i$, $\sigma = -4.01$.

or T could be similar to a Loxodromic Transformation . This implies that σ is not real.

Example : $T(z) = z / (4+i)$, $\sigma = (26 - 32i) / 15$.

Table 2 : Similarity Invariant Behavior Of M Case II (b).

Transformations	Analytic in D	Shrinks C	! Fixed Point
Elliptic	NP	NP	NP
Parabolic	NP	NP	NP
Prop. Hyperbolic	Yes	Yes	Yes
Imp. Hyperbolic	Yes	Yes	Yes
Loxodromic	Yes	Yes	Yes

Upper Triangular Transformations

$$T(z) = (az + b) / d ;$$

I. If $a = d$ then $\sigma = 0$, which implies that T is similar to a Parabolic Transformation.

Example : $T(z) = z + 3$.

II. If $b = d$ then either $\sigma > 0$, which implies that T is similar to a Proper Hyperbolic Transformation,

Example : $T(z) = 4z / 5 + 1$, $\sigma = 1/20$.

or $\sigma \leq -4$, which implies that T is similar to an Improper Hyperbolic Transformation,

Example : $T(z) = (7z - 2) / -2$, $\sigma = -5.79$

or σ is not real , which implies that T is similar to a Loxodromic Transformation.

Example : $T(z) = (4z + i) / i$, $\sigma = (32 + 60i) / -16$.

Table 3: Similarity Invariant Behavior Of U Case (II)

Transformations	Analytic in D	Shrinks C	! Fixed Point
Elliptic	NP	NP	NP
Parabolic	Yes	No	No
Proper Hyperbolic	Yes	Yes	No
Impr. Hyperbolic	Yes	Yes	No
Loxodromic	Yes	Yes	No

III. If a , b , and d are distinct then either $\sigma > 0$, which implies that T is similar to a Proper Hyperbolic Transformation,

Example : $T(z) = (z + 2) / 7$, $\sigma = 5.1$.

or $\sigma \leq -4$, which implies T is similar to an Improper Hyperbolic Transformation,

Example : $T(z) = (7z - 2) / -2$, $\sigma = -5.79$.

or σ is not real, which implies that T is similar to a Loxodromic Transformation.

Example : $T(z) = ((1 + i)z + i) / (5 - 2i)$, $\sigma = (49 - 21i)/58$.

Table 4: Similarity Invariant Behavior Of U Case(III).

Transformations	Analytic in D	Shrinks C	! Fixed Point
Elliptic	NP	NP	NP
Parabolic	NP	NP	NP
Proper Hyperbolic	Yes	Yes	Yes
Impro. Hyperbolic	Yes	Yes	Yes
Loxodromic	Yes	Yes	Yes

Lower Triangular Transformations

$$T(z) = az / (cz + d);$$

I. If $|c| > |d|$ then either, $\sigma = -4$, which implies that T is similar to an Elliptic or Improper Hyperbolic Transformation,

Example: $T(z) = 1 / (2z + 1), \sigma = -4$.

or $\sigma > 0$, which implies that T is similar to a Proper Hyperbolic Transformation ,

Example : $T(z) = 3 / (8z + 1), \sigma = 4.3$.

or σ not real, which implies that T is similar to a Loxodromic Transformation.

Example : $T(z) = -7z / ((2 + i) - i), \sigma = (-14 - 48i) / 7$.

(See Appendix B for image.)

Table 5: Similarity Invariant Behavior Of L Case (I).

Transformations	Analytic in D	Shrinks C	! Fixed Point
Elliptic	No	No	Yes
Parabolic	NP	NP	NP
Proper Hyperbolic	No	No	Yes
Impro. Hyperbolic	No	No	Yes
Loxodromic	No	Yes	No

II. If $|c| < |d|$ then either, $\sigma = -4$ which implies that T is similar to an Elliptic or Improper Hyperbolic Transformation,

Example : $T(z) = (5i)z / (2z + 5i)$, $\sigma = -4$.

or $\sigma > 0$, which implies that T is similar to a Proper Hyperbolic Transformation.

Example: $T(z) = 10z / (z + 6)$, $\sigma = 0.3$.

or $\sigma \leq -4$, which implies that T is similar to an Improper Hyperbolic Transformation.

Example : $T(z) = 10z / (z - 6)$, $\sigma = -4.27$

Or, σ not real which implies T is similar to a Loxodromic Transformation.

Example : $T(z) = iz / ((2 + i)z + 7)$, $\sigma = (-14 - 48i) / 7$.

(See Appendix B for image.)

Table 6: Similarity Invariant Behavior of L Case(II).

Transformations	Analytic in D	Shrinks C	! Fixed Point
Elliptic	Yes	No	Yes
Parabolic	NP	NP	NP
Prop. Hyperbolic	Yes	Yes	No
Impr. Hyperbolic	Yes	Yes	No
Loxodromic	Yes	Yes	Yes

Appendix B

Computer Images

In this appendix, we illustrate computer pictures of, images of the unit disk for, some examples mentioned in this text. In order to better understand the pictures we have used the symbol x and a dot to indicate the location of the singularity point and the fixed points respectively.

These illustrations were generated by letting the transformation act on the set of concentric circles about the origin with radii chosen in increments of a tenth of a unit. Since these circles belong to the hyperbolic pencil of circles with common center at the origin, the images of these circles will also belong to some hyperbolic pencil.

We have used three different colors to indicate the following: 1) Black for the real and imaginary axis 2) Red for the old and new unit circle 3) Purple for the remaining images.

Figure 1 is for $T(z) = (iz) / ((2+i)z + 7)$.

Figure 2 is for $T(z) = (-7z) / ((2+i)z - i)$.

Figure 3 is for $T(z) = (iz) / ((7z) + (2+i))$.

Figure 4 is for $T(z) = (3z) / (2z + 1)$. Although this example is not mentioned in this text, we wanted to include it, in order to demonstrate the image of the disk when the fixed point is on the unit circle.

fig. 1

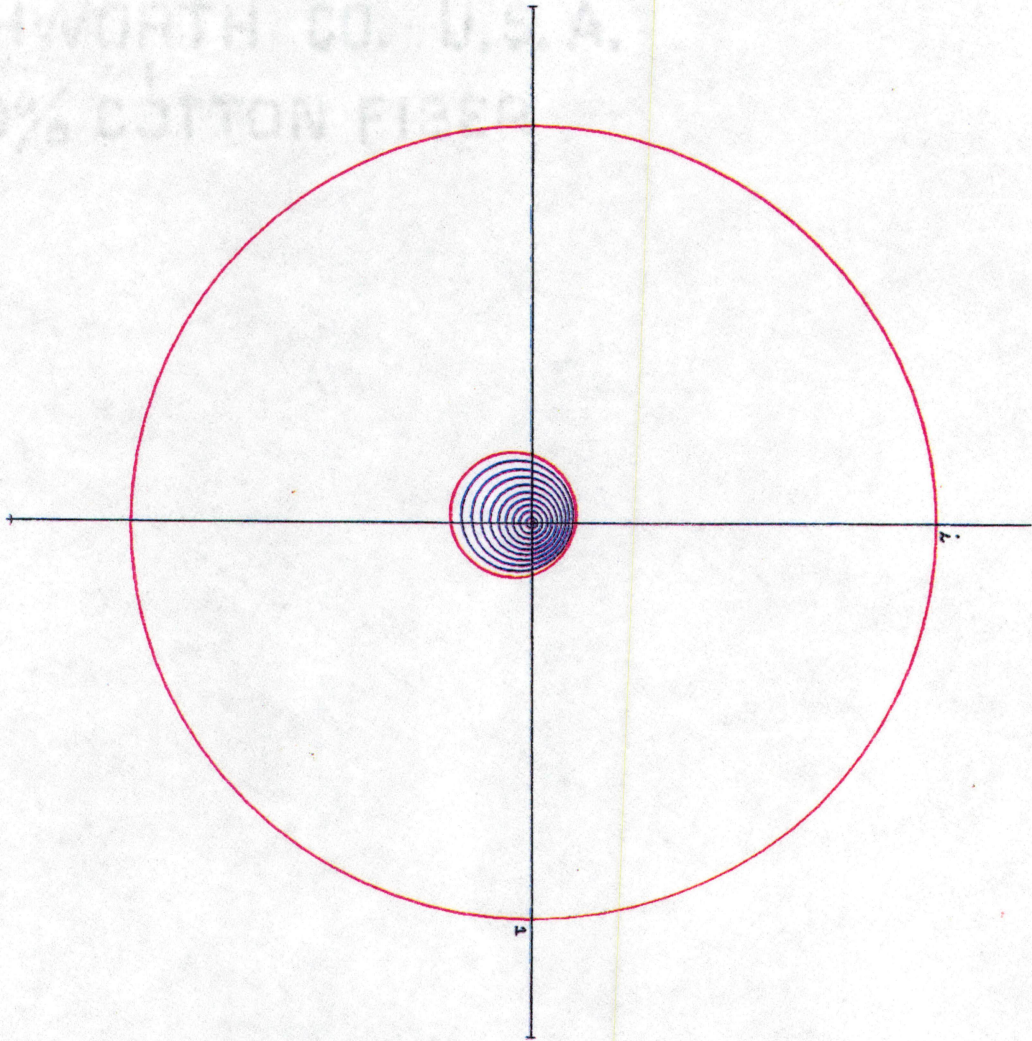


fig. 2

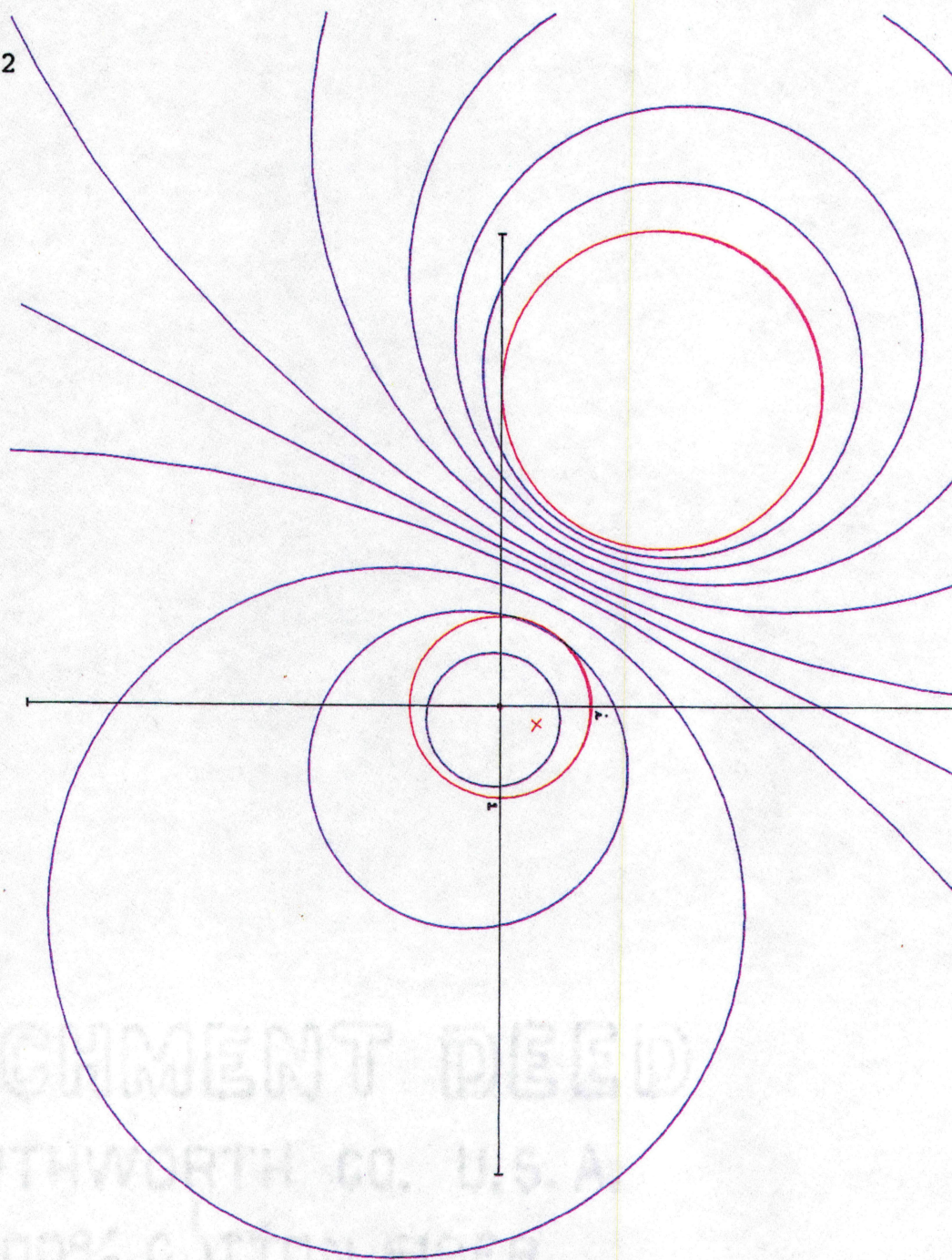


fig. 3

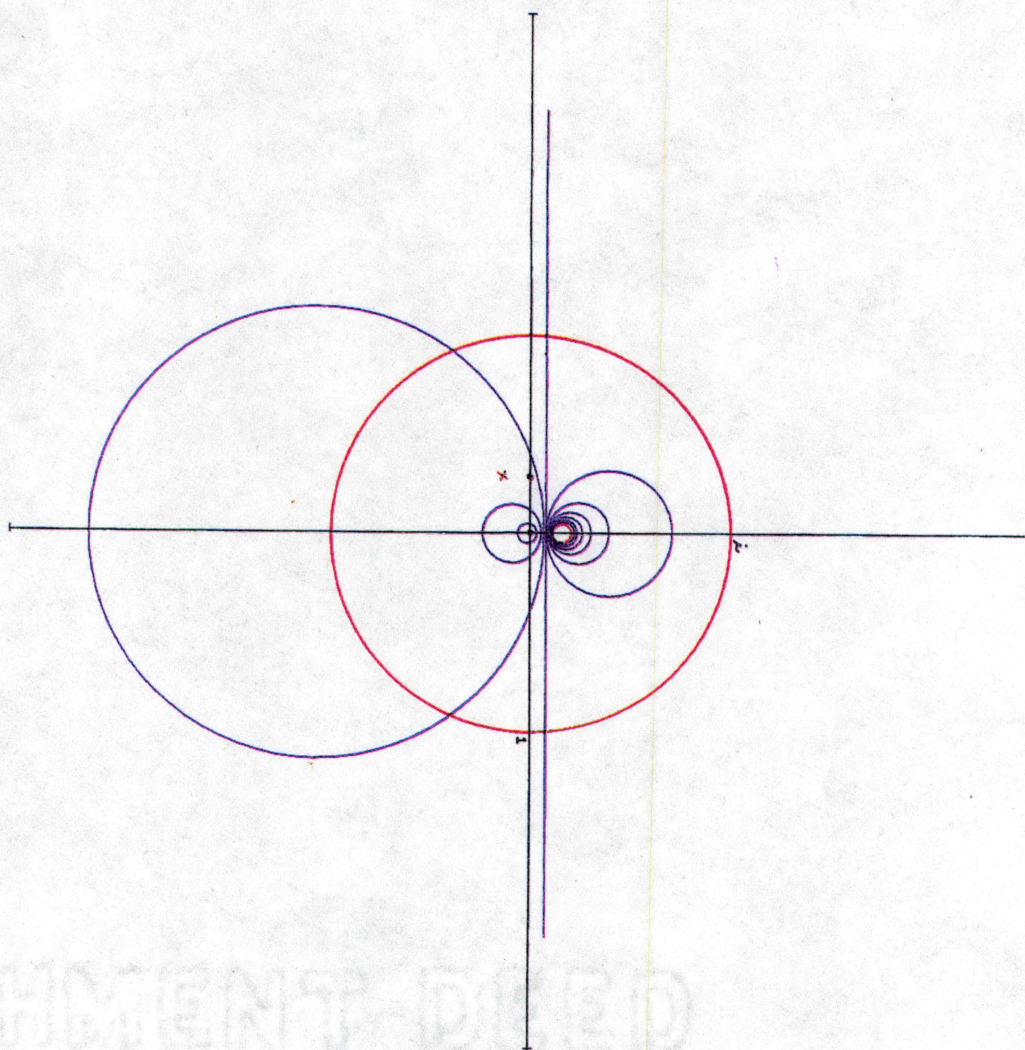
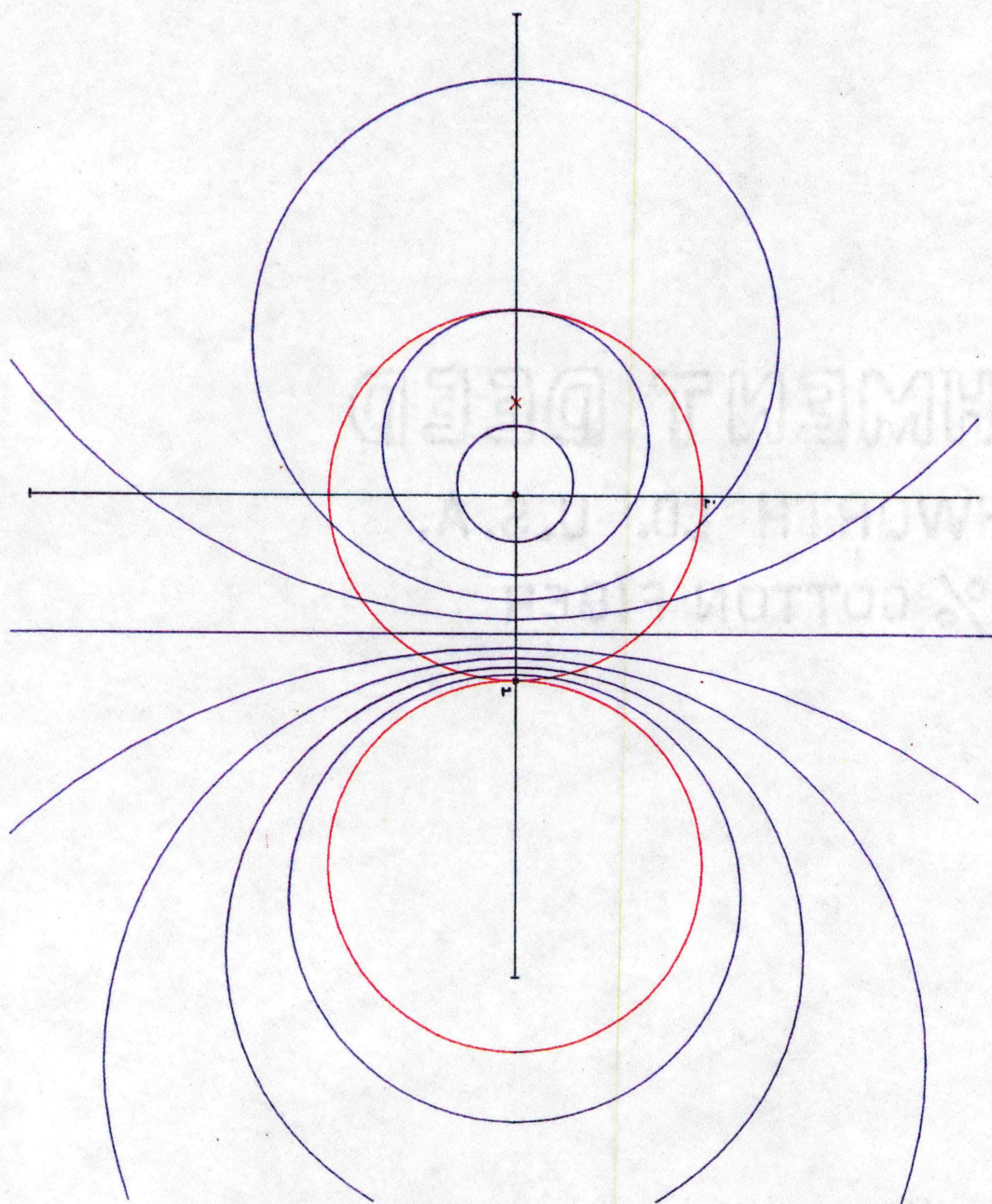


fig. 4



E R R A T U M

Page 36, Appendix C Lemma (3) should read as follows:

Lemma (3) : The group of automorphisms of the unit disk can be factored into the group of rotations about the origin and the group of transformations of the form

$$\begin{bmatrix} 1 & -\alpha \\ -\bar{\alpha} & 1 \end{bmatrix} \quad \text{with } |\alpha| < 1.$$

Appendix C

Supplemental Information

Theorem : Let $T \in L$, with its pole inside D . If T has a fixed point outside of D then T cannot shrink C .

Note: This theorem is in fact true for any linear fractional transformation . That is, given any linear fractional transformation T , whose pole is in D and has fixed point outside D , T cannot shrink C .

The proof of this note requires some techniques beyond the two lemmas used in this paper . But we note that it involves only one additional lemma which is proved in most standard text about linear fractional transformations.

Lemma(3) : The group of automorphisms of the unit disk cannot be factored into the group of rotations about the origin and the group of transformations of the form

$$\begin{bmatrix} 1 & -\alpha \\ -\bar{\alpha} & 1 \end{bmatrix} \quad \text{with } |\alpha| < 1.$$

Using lemma(3) we can extend the above theorem as follows: First note that if $d = 0$ in our transformation then we can assume that $c = 1$. Thus the transformation

$T(z) \neq a + b/z$ has singularity at the origin. It is easy to see that if T has a fixed point outside of D then T cannot shrink the unit circle. On the other hand if $d = 0$ but the pole is inside the disk then we can compose with an automorphism of D and reduce to the case where the pole is at the origin. We will not lose any generality by doing this since the automorphism in lemma(3) leaves the outside of D invariant.

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